TABLE I
ALGORITHMIC PERFORMANCE (cpu s)

Size	kth Best	Grid Search	Penalty Method
(3, 7, 4)	59.23	12.17	8.76
(5, 10, 6)	127.55	33.41	21.85
(6, 14, 8)	111.74	57.29	44.19
(8, 17, 10)	186.23	74.87	74.01
(15, 30, 20)	1200.87	129.33	119.52

Step 1 $a + Kw_0A = a = 1$
$b + Kd = b = 3$
Thus $\operatorname{argMax}\{x + 3y: (x, y) \in Z\} = x(w_0, K = 0) = 10,\ y(w_0, 0) = 14.$

Step 2 $[p - Ax(w_0, 0)]^t = [0, -4, 1, 28, 28]$
Thus $\operatorname{argMin}\{-4w_2 + w_3 + 28w_4 + 28w_5\} = w^*(w_0, 0)^t = (0, 1.5, 0, 0, 0).$

Step 3 $\chi(w_0) = -6$
$w_1^t = (0, 1.5, 0, 0, 0)$
$i = 1$, Go to Step 1.

Iteration 2

Step 1 $a + Kw_1^tA = a = 1$
$b + Kd = b = 3$
Thus $x(w_1, 0) = 10,\ y(w_1, 0) = 14.$

Step 2 remains unchanged

Step 3 $\chi(w_1) = 0$
$\pi\{x(w_1), y(w_1), w_1\} = 36 > 0$
So Optimal solution NOT found.
$K = 0.1$, Go to Step 1.

Iteration 3

Step 1 $a + Kw_1^tA = 1.15$
$b + Kd = 2.7$
$x(w_1, 0.1) = 10,\ y(w_1, 0.1) = 14.$

Step 2 remains unchanged

Step 3 remains unchanged
$K = 0.2$, Go to Step 1.

Iteration 4

Step 1 $a + Kw_1^tA = 1.30$
$b + Kd = 2.40$
$x(w_1, 0.2) = 16,\ y(w_1, 0.2) = 11.$

Step 2 $[p - Ax(w_1, 0.2)]^t = [6, -10, -11, 22, 34]$
Thus $w^*(w_1, 0.2)^t = (0, 0, 3, 0, 0)$

Step 3 $\chi(w_1, 0.2) = -18$
Thus $w_2^t = w^*(w_1, 0.2)^t = (0, 0, 3, 0, 0)$
Go to Step 1.

Iteration 5

Step 1 $a + Kw_2^tA = 2.2$
$b + Kd = 2.4$
$x(w_2, 0.2) = 16,\ y(w_2, 0.2) = 11.$

Step 2 remains unchanged

Step 3 $\chi(w_2, 0.2) = 0.$
$\pi(w_2, 0.2) = 0$
Thus optimality is reached
Optimal solution is $x = 16$, $y = 11$.

The optimal solution from our method is the same for this problem as that obtained by Bialas and Karwan [5]. The total number of iterations, including those required to solve the linear programs was 28.

Table I provides the results of a computational exercise of running 50 randomly selected problems of five different sizes for each method. The size of the problem is denoted (n_1, n_2, m) where n_1 and n_2 stand for the dimension of the leader's and follower's decision vectors, respectively, and m is the number of constraints. The problems were run on an ATT PC6300+ microcomputer with Intel 80286 microprocessor, and 80287 math coprocessor. Each linear program was solved using the LINDO package, and Pascal programs linked the different components of the iteration together. The penalty function method proposed in this note marginally outperformed Bard's [2] grid search method and easily outperformed the kth best method [5]. In a companion paper [14], we provide an algorithm to find global optimal solutions for these problems.

ACKNOWLEDGMENT

The authors thank, without implicating, K. Jornsten, J. Judice, M. Karwan, H. Sherali, and two anonymous referees for comments and suggestions. They also thank R. Nigam for computer programming assistance.

REFERENCES

[1] E. Aiyoshi and K. Shimizu, "A solution method for the static constrained Stackelberg problem via penalty method," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 1111–1114, Dec. 1984.
[2] J. F. Bard, "An efficient point algorithm for a linear two-stage optimization problem," *Operations Res.*, pp. 670–684, July–Aug. 1983.
[3] M. Bazaraa and C. M. Shetty, *Nonlinear Programming: Theory and Algorithms*. New York: Wiley, 1979.
[4] W. F. Bialas and M. H. Karwan, "On two-level optimization," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 211–214, 1982.
[5] —, "Two-level linear programming," *Management Sci.*, vol. 30, no. 8, pp. 1004–1020, Aug. 1984.
[6] W. Candler and R. Townsley, "A linear two-level programming problem," *Comput. Operations Res.*, vol. 9, no. 1, pp. 59–76, 1982.
[7] T.-S. Chang and Y. C. Ho, "Stochastic Stackelberg games: Nonnested multistage multiagent incentive problems," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 477–488, Apr. 1983.
[8] Y. C. Ho, P. Luh, and R. Muralidharan, "Information structures, Stackelberg games, and incentive controllability," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 454–460, Feb. 1981.
[9] P. B. Luh, S.-C. Chang, and T.-S. Chang, "Solutions and properties of multistage Stackelberg games," *Automatica*, vol. 20, no. 2, pp. 251–256, 1984.
[10] G. P. Papavassilopoulos and J. B. Cruz, "Nonclassical control problems and Stackelberg games," *IEEE Trans. Automat. Contr.*, vol. AC-24, no. 2, pp. 155–166, Feb. 1979.
[11] K. Shimizu and E. Aiyoshi, "A new computational method for Stackelberg and mini-max problems by the use of a penalty method," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 2, pp. 460–466, Feb. 1981.
[12] M. Simaan and J. B. Cruz, "On the Stackelberg strategy in nonzero-sum games," *J. Optimiz. Theory Appl.*, vol. 11, no. 5, pp. 533–555, 1973.
[13] B. Tolwinski, "Closed-loop Stackelberg solution to multistage linear quadratic games," *J. Optimiz. Theory Appl.*, vol. 34, no. 4, pp. 485–501, 1981.
[14] D. J. White and G. Anandalingam, "A penalty function approach for solving bi-level linear programs," Dep. Syst., Univ. Pennsylvania, Philadelphia, working paper, Nov. 1989.

Application of Lyapunov Functionals to Studying Stability of Linear Hyperbolic Systems

MARIUSZ ZIÓŁKO

Abstract—**The Lyapunov functional method is used to prove the stability conditions for Cauchy problems and initial-boundary value problems if the system is described by a set of linear first-order partial differential equations of the hyperbolic type. Although the considered system is linear, it is possible to obtain necessary and sufficient conditions for stability only when matrices in differential equations and boundary conditions have some special properties.**

I. INTRODUCTION

The stability of a system described by a set of first-order partial differential equations of hyperbolic type is an interesting problem in control theory and arises in stability theory of numerical methods for hyperbolic systems. Gunzburger [2] considered the stability of initial-boundary value problems for a system consisting of two equations. The application of the Lyapunov functional method to stability of linear hyperbolic systems with more than two equations leads to the searching for functionals with diagonal matrices. The questions of whether or not there exists a positive

Manuscript received March 24, 1989; revised September 29, 1989.
The author is with the Institute of Automatic Control, Technical University of Mining and Metallurgy, 30-058 Kraków, Poland.
IEEE Log Number 9036823.

diagonal matrix G such that $D^TG+GD<0$ or $S^TGS-G<0$, do not have a simple answer. The characterization of the class of matrices D and S which have these properties is either a set of sufficient conditions or a set of necessary conditions. Although the Khalil [4] algorithm gives a definite answer to the first question, it is difficult to use this algorithm, especially if matrix D has high dimension. These difficulties enable us to give necessary and sufficient conditions for stability of linear hyperbolic systems only in some special cases.

II. The Hyperbolic Systems

Consider the canonical form of this linear hyperbolic system consisting of first-order partial differential equations

$$\frac{\partial y}{\partial t}+\Lambda\frac{\partial y}{\partial x}=Dy \tag{1}$$

where Λ and D are $n\times n$ real and constant matrices. Without loss of generality, Λ is taken to be the diagonal form

$$\Lambda=\begin{pmatrix}\Lambda^- & 0\\ 0 & \Lambda^+\end{pmatrix}\qquad \Lambda^-<0\qquad \Lambda^+<0 \tag{2}$$

in such a way that elements on the diagonal fulfill the inequalities

$$+\infty<\lambda_1\le\lambda_2\le\cdots\le\lambda_p<0<\lambda_{p+1}\le\lambda_{p+2}\le\cdots\le\lambda_n<\infty. \tag{3}$$

If we prescribe C^1 continuous initial values

$$y(x,0)=y_0(x)\qquad 0\le x\le 1 \tag{4}$$

the solution $y\colon \Psi\to R^n$ of Cauchy problem (1), (4) is uniquely (cf. [1]) determined for

$$\Psi=\{(x,t)\colon 0\le t\le 1/(\lambda_n-\lambda_i),\ \lambda_n t\le x\le 1+\lambda_1 t\}. \tag{5}$$

Additionally, we can prescribe boundary conditions

$$\begin{aligned}y^+(0,t)&=S_0y^-(0,t)\\ y^-(1,t)&=S_1y^+(1,t)\qquad 0\le t\le T\end{aligned} \tag{6}$$

where the unknown vector $y(x,t)\in R^n$ is divided into two parts: $y^-(x,t)\in R^p$ and $y^+(x,t)\in R^{n-p}$ corresponding to the partition of Λ, and S_0, S_1 are fixed $(n-p)*p$ and $p*(n-p)$ matrices, respectively. The solution of initial-boundary value problem (1), (4), (6) is uniquely (cf. [1]) determined for

$$\Psi=\{(x,t)\colon 0\le x\le 1,\ 0\le t\le T\}. \tag{7}$$

III. The Energy Functional for Hyperbolic Systems

The linear homogeneous dynamic system (1) is asymptotically stable if $E(t)=\|y\|^2\to 0$ as $t\to\infty$. The main problem consists of finding a monotonic norm. It is well known that for a system described by a set of ordinary differential equations

$$\frac{dy}{dt}=Dy \tag{8}$$

we can always find constant positive defined matrices G such that the energy functional

$$E(t)=\|y\|_0^2=y^TGy \tag{9}$$

is monotonic.

The energy functional for the system (1) can be defined as

$$E_{(t)}=\|y\|_0^2=\int_{x_b}^{x_e}y^TGy\,dx \tag{10}$$

where G is a fixed positive definite $n\times n$ matrix. For Cauchy problem (1), (4), we take

$$x_b(t)=\lambda_n t\qquad x_e(t)=1+\lambda_1 t, \tag{11}$$

while for the initial-boundary value problem (1), (4), (6)

$$x_b(t)=0\qquad x_e(t)=1. \tag{12}$$

The time rate of change of Lyapunov functions for initial-boundary value problem (1), (4), (6) is given by

$$\frac{dE}{dt}=\int_0^1 y^T(D^TG+GD)y\,dx-y^T\Lambda Gy|_0^1+\int_0^1 y^T(\Lambda G-G\Lambda)\frac{\partial y}{\partial x}\,dx. \tag{13}$$

The first derivative of the energy functional for the Cauchy problem is obtained by using the Leibnitz pattern. Taking into account the equation (1) and integrating by parts, we finally obtain

$$\begin{aligned}\frac{dE}{dt}&=y^T(1+\lambda_1t,t)[\lambda_iI-\Lambda]Gy(1+\lambda_1t,t)\\ &\quad+y^T(\lambda_nt,t)[\Lambda-\lambda_nI]Gy(\lambda_nt,t)\\ &\quad+\int_{\lambda_nt}^{1+\lambda_1t}y^T(D^TG+GD)y\,dx\\ &\quad+\int_{\lambda_nt}^{1+\Lambda_1t}y^T(\Lambda G-G\Lambda)\frac{\partial y}{\partial x}\,dx.\end{aligned} \tag{14}$$

To establish the sign of dE/dt we must make an additional assumption. If matrix G is diagonal, then matrices Λ and G commute, i.e., $G\Lambda=\Lambda G$, and the last bilinear segments of (13) and (14) vanish. This restriction for matrix G makes certain that the necessary and sufficient conditions for asymptotic stability can be obtained if we assume a certain "symmetry" of the boundary value problem.

Taking into account that $G\Lambda=\Lambda G$ and substituting boundary conditions (6) to (13), we obtain

$$\frac{dE}{dt}=\frac{dE^i}{dt}+\frac{dE^b}{dt} \tag{15}$$

where

$$\frac{dE^i}{dt}=\int_0^1 y^T(D^TG+GD)y\,dx \tag{16}$$

$$\begin{aligned}\frac{dE^b}{dt}&=y^{-T}(0,t)[G^-\Lambda+S_0^TG^+\Lambda^+S_0]y^-(0,t)\\ &\quad-y^{+T}(1,t)[G^+\Lambda^++S_1^TG^-\Lambda^-S_1]y^+(1,t).\end{aligned} \tag{17}$$

G^- and G^+ are the partitions of G, similarly, like Λ^- and Λ^+, are the partitions of matrix Λ.

The time rate of change of the energy functional (15) is divided into two parts: (16) and (17). The first part depends on matrix D and the second depends on matrices S_0 and S_1. Matrix D is the coefficient of differential equation (1). This equation describes the dynamics of a system for all x which belong to the interior of segment [0, 1]. That is why, if there exists a positive definite diagonal matrix G such that the inequality $dE^i/dt<0$ holds for every $t\ge 0$ and every nonzero initial condition, we call the initial-boundary value problem (1), (4), (6) asymptotically interior stable. Matrices S_0 and S_1 describe the properties of the solution for the boundaries of segment $x\in[0,1]$. For this reason, we call the problem (1), (4), (6) asymptotically boundary stable if there exists a positive definite diagonal matrix G such that the inequality $dE^b/dt<0$ holds.

IV. Interior Stability of the Initial-Boundary Value Problem

For the reasons mentioned above, we must limit our considerations to the cases when matrix G in functional (10) is diagonal. It is easy to notice that, if there exists a diagonal positive matrix G such that $dE^i/dt < 0$ for all $t \geq 0$ and arbitrary $y_0(x) \neq 0$, then $D^T G + GD < 0$. This quadratic form is negative definite for positive matrix G only if all real parts of eigenvalues of matrix D are negative. It is also easy to verify that if all eigenvalues of matrix $D + D^T$ are negative, then we can take G equal to the unit matrix and obtain $dE^i/dt < 0$ for every $y \neq 0$. If we assume, additionally, that for this case, matrix D is symmetric, then we obtain not only sufficient but also necessary conditions for interior stability. This idea can be developed for a much larger class of matrices D.

Theorem 1: If matrix D is symmetrizable by a diagonal positive definite matrix, e.g., $K^2 D = D^T K^2$, then initial-boundary value problem (1), (4), (6) is interior stable if and only if all eigenvalues of matrix D are negative.

Proof: For a symmetrizable matrix D we have $KDK^{-1} = K^{-1}D^T K$. It follows that KDK^{-1} is a symmetric matrix and all eigenvalues of D are real. Substituting $G = K^2$ and $y = K^{-1}\eta$, we obtain

$$y^T(GD + D^T G)y = 2\eta^T KDK^{-1}\eta. \tag{18}$$

This quadratic form is negative definite if and only if all eigenvalues of matrix D are negative. ■

V. Boundary Stability of a Mixed Problem

Matrices S_0 and S_1 determine the influence of boundary conditions for a solution of hyperbolic equations. In particular, the boundary wave reflections are described by these matrices. Reflections which increase energy can provide instabilization of the system. Gunzburger and Plemmons [3] have presented necessary and sufficient conditions for the matrices S_0 and S_1 for the energy conserving norm, i.e., $dE^b/dt = 0$.

The influence of boundary conditions for the time rate of change of Lyapunov functional (10) is given by (15), (17). The hyperbolic system is boundary stable if there exists a diagonal and positive definite matrix G such that $dE^b/dt < 0$ for every $t \geq 0$ and arbitrary y not equal to zero simultaneously for both boundaries. It follows from this definition that for a boundary stable system, two inequalities for quadratic forms are fulfilled

$$\begin{aligned} S_0^T G^+ \Lambda^+ S_0 &< -G^- \Lambda^- \\ S_1^T(-G^-\Lambda^-)S_1 &< G^+\Lambda^+. \end{aligned} \tag{19}$$

Substituting $-G^-\Lambda^-$ for $S_0^T G^+ \Lambda^+ S_0$ into the second inequality, we obtain

$$(S_0 S_1)^T G^+ \Lambda^+ S_0 S_1 - G^+ \Lambda^+ < 0. \tag{20}$$

If we take for a discrete system

$$y_{n+1} = S_0 S_1 y_n, \tag{21}$$

the energy functional

$$E_n = y_n^T G^+ \Lambda^+ y_n, \tag{22}$$

then inequality (20) is equivalent with inequality

$$E_{n+1} - E_n < 0. \tag{23}$$

This remark has a simple physical interpretation. The disturbances traveling between edges reflect from boundaries at discrete times. The matrix product $S_0 S_1$ is a composition of two reflections: first, from boundary $x = 1$ and next, from boundary $x = 0$. It is well known that inequality (20) can be fulfilled by a positive definite matrix $G^+\Lambda^+$ only if all modules of eigenvalues of matrix $S_0 S_1$ are less than 1. We obtain the identical conclusion for matrix product $S_1 S_0$. This case denotes reflection from boundary $x = 0$, at first, and then from boundary $x = 1$.

Theorem 2: If the boundary conditions (6) fulfill the postulate of symmetribility of matrix

$$S = \begin{pmatrix} 0 & S_1 \\ S_0 & 0 \end{pmatrix} \tag{24}$$

by a diagonal positive definite matrix, then initial-boundary value problem (1), (4), (6) is asymptotic boundary stable, if and only if all eigenvalues of matrix $S_0 S_1$ or equivalently, $S_1 S_0$, are less than 1.

Proof: Condition (17) for the boundary stability can be written in matrix notation

$$y^T(S^T|G\Lambda|S - |G\Lambda|)y < 0 \tag{25}$$

where $y = [y^{-T}(0, t) \quad y^{+T}(1, t)]^T$ and $|\cdot|$ denotes the absolute value of all elements of matrix, i.e.,

$$|G\Lambda| = \begin{pmatrix} -G^-\Lambda^- & 0 \\ 0 & G^+\Lambda^+ \end{pmatrix}.$$

For a symmetrizable matrix S we have $KSK^{-1} = K^{-1}S^T K$, that is KSK^{-1} is a symmetric matrix. It follows that all eigenvalues of matrix S are real. Substituting $|G\Lambda| = K^2$ and $\eta = Ky$, we obtain condition (25) for the boundary stability in the form

$$\frac{\eta^T(KSK^{-1})^2\eta}{\eta^T\eta} < 1. \tag{26}$$

This Rayleigh's quotient is less than 1 for arbitrary $\eta \neq 0$, if and only if, all modules of eigenvalues of the matrix S are less than 1. If λ denotes eigenvalue of matrix S, and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is its block eigenvector, then

$$\begin{pmatrix} -\lambda I & S_1 \\ S_0 & -\lambda I \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0. \tag{27}$$

It follows that

$$\begin{aligned} S_0 S_1 y_2 &= \lambda^2 y_2 \\ S_1 S_0 y_1 &= \lambda^2 y_1, \end{aligned} \tag{28}$$

i.e., λ^2 is an eigenvalue of matrices $S_0 S_1$ and $S_1 S_0$. ■

Symmetribility of matrix S means that there are positive definite diagonal matrices G^+ and G^- such that

$$S_0^T G^+ \Lambda^+ = -G^- \Lambda^- S_1. \tag{29}$$

This condition can be fulfilled only if each element of matrix S_0^T and element of matrix S_1 situated in the same place have common properties. They must have the same sign or must be both equal to zero. Such a restriction is not fulfilled by a great number of boundary conditions for equations of practical importance. For these cases, it is possible to formulate only sufficient conditions for boundary stability.

Theorem 3: The mixed problem (1), (4), (6) is boundary stable if there exists a diagonal nonsingular matrix K such that $KS = S^T K$ and all modules of eigenvalues of matrix $K^{-1}|K|S$, or equivalently $S|K|K^{-1}$, are less than 1.

Proof: It is always possible to find such a diagonal matrix J that $|K| = JKJ$. Elements of matrix J have real value 1 or imaginary value $-j$. The matrices $K^{-0.5}$ and J are complex, but their product $K^{-0.5}J$ is a real diagonal matrix. If $|G\Lambda| = |K|$ and $y = K^{-0.5}J\eta$, then the condition (25) for the boundary stability takes the form

$$\eta^T(K^{-0.5}JS^T JKJSJK^{-0.5} - I)\eta < 0. \tag{30}$$

It results from assumption $KS = S^T K$ that matrix $K^{-0.5}JSJK^{-0.5}$ is symmetric. Therefore, inequality (30) is satisfied for every nonzero real vector η if all modules of eigenvalues (which are real) of a complex matrix JSJ are less than 1. Now, if y denotes an eigenvector of matrix JSJ and λ is its eigenvalue, then $(JSJ - \lambda I)y = 0$. It follows that $(J^2S - \lambda I)Jy = 0$ and $(SJ^2 - \lambda I)J^{-1}y = 0$; that is, matrices JSJ and $J^2S = K^{-1}|K|S$ and $SJ^2 = S|K|K^{-1}$ have the same eigenvalues. ■

VI. Stability of the Cauchy Problem

Under the assumption that matrix G is diagonal, the first derivative (14) of the energy functional for Cauchy problem (1), (4) takes the form

$$\begin{aligned} \frac{dE}{dt} &= y^T(1 + \lambda_1 t, t)[\lambda_1 I - \Lambda]Gy(1 + \lambda_1 t, t) \\ &\quad + y^T(\lambda_n t, t)[\Lambda - \lambda_n I]Gy(\lambda_n, t) \\ &\quad + \int_{\lambda_n t}^{1+\lambda_1 t} y^T(D^T G + GD)y\, dx. \end{aligned} \tag{31}$$

The diagonal matrices $(\lambda_1 I - \Lambda)G$ and $(\Lambda - \lambda_n I)G$ have one element on the diagonal equal to zero and the other elements are negative.

Theorem 4: The stability of the Cauchy problem is equivalent to the interior stability for the initial-boundary value problem.

Proof: Matrices $\lambda_1 I - \Lambda$ and $\Lambda - \lambda_n I$ are negative semidefinite. It makes that negative definite matrix $D^T G + GD$ creates the sufficient condition for the stability of the Cauchy problem, similarly, like it is for the interior stability.

The necessary conditions are also identical for both problems. If the Cauchy problem is stable, then $dE/dt < 0$ for all nonzero initial values. It also must deal with all initial conditions for which the two first quadratic forms in (31) are equal to zero. Such initial conditions exist and can be determined if the solutions along the characteristic defined by λ_1 and λ_n are assumed. $y_1(1+\lambda_1 t, t)$ and $y_n(\lambda_n t, t)$ can be taken arbitrarily and $y_2(1+\lambda_1 t, t) = y_3(1+\lambda_1 t, t) = \cdots = y_n(1+\lambda_1 t, t) = y_1(\lambda_n t, t) = y_2(\lambda_n t, t) = \cdots = y_{n-1}(\lambda_n t, t) = 0$. It follows that matrix $D^T G + GD$ must be negative definite. ■

VII. Example

Consider the distributed electric network described by the equation

$$\begin{pmatrix} \frac{\partial i}{\partial t} \\ \frac{\partial u}{\partial t} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial i}{\partial x} \\ \frac{\partial u}{\partial x} \end{pmatrix} = \begin{pmatrix} -\frac{R}{L} & 0 \\ 0 & -\frac{F}{C} \end{pmatrix} \begin{pmatrix} i \\ u \end{pmatrix} \tag{32}$$

where i denotes current and u voltage. R, L, F, C are distributed parameters of the network (resistance, inductance, conductance, and capacitance). The boundary conditions are described by the following equations:

$$(R_1 \quad 1) \begin{pmatrix} i_{(0,t)} \\ u_{(0,t)} \end{pmatrix} = 0 \tag{33}$$

$$(R_2 \quad -1) \begin{pmatrix} i_{(1,t)} \\ u_{(1,t)} \end{pmatrix} = 0 \tag{34}$$

where R_1, R_2 are resistances at the boundaries of the network. By transformation

$$\begin{pmatrix} i \\ u \end{pmatrix} = \begin{pmatrix} \sqrt{C} & \sqrt{C} \\ -\sqrt{L} & \sqrt{L} \end{pmatrix} y \tag{35}$$

the considered system can be taken to canonical form

$$\frac{\partial y}{\partial t} + \begin{pmatrix} -\frac{1}{\sqrt{LC}} & \\ & \frac{1}{\sqrt{LC}} \end{pmatrix} \frac{\partial y}{\partial x} = \frac{1}{2} \begin{pmatrix} -\frac{R}{L} - \frac{F}{C} & \frac{F}{C} - \frac{R}{L} \\ \frac{F}{C} - \frac{R}{L} & -\frac{R}{L} - \frac{F}{C} \end{pmatrix} y \tag{36}$$

$$y(x, 0) = \frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{C}} & -\frac{1}{\sqrt{L}} \\ \frac{1}{\sqrt{C}} & \frac{1}{\sqrt{L}} \end{pmatrix} \begin{pmatrix} i_0 \\ u_0 \end{pmatrix} \tag{37}$$

$$y^+(0, t) = \frac{\sqrt{L} - R_i\sqrt{C}}{R_i\sqrt{C} + \sqrt{L}} y^-(0, t) \tag{38}$$

$$y^-(1, t) = \frac{\sqrt{L} - R_2\sqrt{C}}{R_2\sqrt{C} + \sqrt{L}} y^+(1, t). \tag{39}$$

The eigenvalues of matrix D are negative: $d_1 = -R/L$, $d_2 = -F/C$. It means that the system is interior stable.

The considered system consists of only two differential equations. It is easy [5] to find the Lyapunov functional (10) for such systems. Diagonal matrix G has the general form

$$G = \begin{pmatrix} 1 & \\ & g \end{pmatrix}. \tag{40}$$

Only if parameter g satisfies inequalities

$$\frac{(\sqrt{RC} - \sqrt{FL})^2}{(\sqrt{RC} + \sqrt{FL})^2} < g < \frac{(\sqrt{RC} + \sqrt{FL})^2}{(\sqrt{RC} - \sqrt{FL})^2} \tag{41}$$

then $dE^i/dt < 0$ holds at each time t for every $y_0 \neq 0$.

The system under consideration is also boundary stable because

$$|S_0 S_1| = \left| \frac{\sqrt{L} - R_1\sqrt{C}}{R_1\sqrt{C} + \sqrt{L}} \cdot \frac{\sqrt{L} - R_2\sqrt{C}}{R_2\sqrt{C} + \sqrt{L}} \right| < 1. \tag{42}$$

The condition $dE^b/dt < 0$ is fulfilled only if parameter g takes values

$$\left(\frac{\sqrt{L} - R_2\sqrt{C}}{R_2\sqrt{C} + \sqrt{L}} \right)^2 < g < \left(\frac{R_1\sqrt{C} + \sqrt{L}}{\sqrt{L} - R_1\sqrt{L}} \right)^2. \tag{43}$$

From inequalities (41) and (43), we conclude that it is possible to find g such that $dE^b/dt < 0$ and $dE^i/dt < 0$ hold at each time, simultaneously. This property does not occur in general.

References

[1] R. Courant and D. Hilbert, *Methods of Mathematical Physics Vol. II*; and R. Courant, *Partial Differential Equations*. New York: Interscience, 1962, pp. 449, 461–475, 668–676.

[2] M. D. Gunzburger, "On the stability of Galerkin methods for initial-boundary value problems for hyperbolic systems," *Math. Comput.*, vol. 31, pp. 661–675, 1977.

[3] M. D. Gunzburger and R. J. Plemmons, "Energy coserving norms for the solution of hyperbolic systems of partial differential equations," *Math. Comput.*, vol. 33, pp. 1-10, 1979.

[4] H. K. Khalil, "On the existence of positive diagonal P such that $PA + A^T P < 0$," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 181–184, 1982.

[5] M. Ziółko, "Stability of initial-boundary value problem for hyperbolic systems," *Notes Numer. Fluid Mechan.*, vol. 24, pp. 698–707, 1988.

Relationship Between the Trace and Maximum Eigenvalue Norms for Linear Quadratic Control Design

M. J. GRIMBLE

***Abstract*—The use of a sum of squares H_2 norm is considered for optimal control system design. The relative advantages of the LQG (trace norm) and the H_∞ (sup norm) cost functions are discussed. The relationship between the resulting solutions is established and it is shown that an LQG controller also minimizes a sum of squares norm for a system with larger disturbances.**

Notation and Mathematical Preliminaries

λ_i, σ_i	Eigenvalues, singular values.
$\lambda_{max}, \sigma_{max}$	Maximum eigenvalue, maximum singular value.
$R^{r\times m}(z^{-1})$	$r \times m$ rational transfer-function matrix in z^{-1}.

Manuscript received April 10, 1989; revised September 22, 1989.

The author is with the Industrial Control Unit, University of Strathclyde, Glasgow G1 1QE, Scotland, U.K.

IEEE Log Number 9036824.