DESIGN OF INTEGER FILTERS FOR TRANSMULTIPLEXER PERFECT RECONSTRUCTION

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ABSTRACT

A new and efficient method of designing the transmultiplexer filters is presented. The bilinear equations posed for the FIR filters are solved to achieve perfect reconstruction. For a given combining filter bank a separation filter bank can be developed by solving a set of algebraic equations. Some examples of a two-channel and four-channel transmultiplexer system are provided to illustrate the method of filter bank designing. Due to the incorporation of integer filters a perfect reconstruction was realized and crosstalk was completely eliminated not only theoretically but also in practice. Such transmultiplexers can be applied to coded signals transmission. It is pointed that the orthogonal filters can be obtained as well.

1. INTRODUCTION

Frequency-Division Multiplexing (FDM) is an important method of combining signals of several users into one signal for the transmission by a single channel in currently used telecommunications systems. However, FDM has some disadvantages, one of the main problems is a very strong dependence of users data and quality on one frequency subband. Interferences and noises occur in a narrow band, especially in radio communication. It implies a break of transmission for one or two users. It is much more convenient for telecommunications companies to spread that disturbances onto many users in the way that there will be no breaks and only a small loss of quality. Such idea was implemented in FDM by frequency hopping, but the new method with users bands independent from a single frequency is still needed. Probably a currently introduced method – the Code Division Multiple-Access (CDMA) [6] system has fulfilled that important expectation. The frequency of the transmitted signal is then made to vary according to a defined pattern (code). The spectrum of each user's signal is spread over the whole channel bandwidth. It can be only intercepted by a receiver whose frequency response is programmed with the same code. The narrowband data users signal is multiplied by a spreading signal - a very large bandwidth signal. All users in a CDMA system may transmit simultaneously by using the same carrier frequency. Different users' spreading signals are approximately orthogonal to each other. The receiver performs a correlation to detect data addressed to a given user, while signals from all the other users appear as noise. The receiver needs the spreading signal used in the transmitter for detecting. The system works with uncoordinated transmission - users have no knowledge of the other users [7].

Transmultiplexers [4] have the same advantage, called generally the "spread spectrum". According to their mathematical models, transmultiplexers are much more general systems. All other methods of multiple-access are certain realizations of transmultiplexers with suitable banks of filters chosen in a specific way. The greatest challenge is to find the required filters. This is not an easy task although the necessary conditions given in the *z*-domain are well known [5]. An innovative and efficient method of designing such filters is suggested in this paper.

In many applications there has been a growing interest in reversible integer-to-integer filter banks. Signals are then invertible in finite-precision arithmetic and map integers to integers. Due to this property, transmultiplexers of this type have additional advantages: they can be applied to transmitting lossless compressed signals, minimal memory can be used and complexity of computations can be low. Due to properties like these, there is a clear need to consider reversible integer-to-integer filter banks. A particularly interesting case appears when filters are orthogonal. Orthogonality is the fundamental concept in the design of many communication systems [1].

2. TRANSMULTIPLEXERS

Each transmultiplexer combines several signals into a single signal. The transmultiplexers were originally studied in the context of converting Time Division Multiplexing (TDM) into FDM. Their main application is for simultaneous transmission of several data signals through a single channel.

Fig.1 shows the classical schematic diagram of the four-channel transmultiplexer. At the transmitter, the M input signals were upsampled, filtered and summed to obtain a composite signal. This composite signal can be transmitted over a single transmission channel. At the receiver end, the composite signal is relayed to the four channels of the separation part, where the signal is filtered and downsampled to recover the original input signals. The system presented in

Fig.1 consists of linear and time-invariant elements. This facilitates the construction of a mathematical model for such systems.

The basic idea is the reversibility of all procedures of transmultiplexation in such a way that all output signals could be recovered as precisely as possible. For the well-designed transmultiplexers, the output signal s_i^{out} approximates the input signal s_i^{in}, where i is a signal number, $i \in \{1,2,\ldots,M\}$. The main task in a transmultiplexer system design is to develop an appropriate separation filter bank such that the output signals can resemble the original input signals as much as possible. A transmultiplexer achieves perfect reconstruction if s_i^{out} is only a delayed version of s_i^{in}, namely if there exist a positive integer τ such that

$$s_i^{\text{out}}(n) = s_i^{\text{in}}(n-\tau). \qquad (1)$$

The dependence of output s_i^{out} from inputs s_k^{in}, where $i,k \in \{1,2,\cdots,M\}$ is described [4] in the z-transform domain by equation

$$\bar{s}_i^{\,\text{out}}(z) = \frac{1}{M}\sum_{k=1}^{M}\bar{s}_k^{\,\text{in}}(z)\left[\sum_{m=0}^{M-1} H_k^{\,c}(w_M^m z^{1/M}) H_i^{\,s}(w_M^m z^{1/M})\right], \qquad (2)$$

where $w_M = e^{-2\pi\cdot j/M}$ and $\bar{s}_i(z)$, $H^c(z)$, $H^s(z)$ stand for the z-spectrum of signal $s_i(n)$, combining and separation transfer functions, respectively. It means that each output signal depends on all inputs signals. A key point is that the constituent signals should be recoverable from the combined signal. To fulfill the perfect reconstruction condition (1), we obtain a set of M^2 equations

$$\frac{1}{M}\sum_{m=0}^{M-1} H_k^c\left(w_M^m z^{1/M}\right) H_i^s\left(w_M^m z^{1/M}\right) = z^{-\tau}\delta_{k,i} \qquad (3)$$

from (2), where $\delta_{k,i}$ is a Kronecker function. In that way necessary and sufficient conditions for a crosstalk-free transmultiplexer were obtained [5].

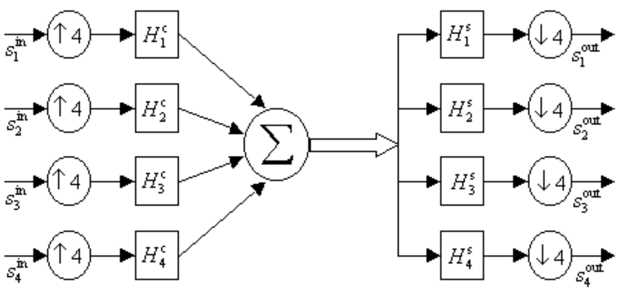

Figure 1. A scheme of 4-channel transmultiplexer.

Under assumption that all filters are FIR type of order I

$$H(z) = \sum_{i=0}^{I} h(i)\, z^{-i},$$

conditions (3) are equivalent to

$$\sum_{n=0}^{[2I/M]} z^{-n} \sum_{l=\max\{0,Mn-I\}}^{\min\{Mn,I\}} h_k^c(Mn-l)\, h_i^s(l) = z^{-\tau}\delta_{k,i}\ , \qquad (4)$$

where $h_i^s(l)$ means the l-th coefficient of filter $H_i^s(z)$, $i \in \{1,2,\ldots,M\}$. Similarly $h_k^c(Mn-l)$ stands for the coefficients of the $H_k^c(z)$ filter. The assumption that all filters are of order I gives us restriction of sum (4), i.e. it gives an additional condition $0 \le Mn - l \le I$, which is equivalent to $\max\{0, Mn-I\} \le l \le \min\{Mn, I\}$. Second inequality is used to calculate the summation range of equations (4).

Equations (4) are fulfilled for all $z \in C$ if and only if

$$\sum_{l=\max\{0,Mn-I\}}^{\min\{Mn,I\}} h_k^c(Mn-l)\, h_i^s(l) = \delta_{k,i}\delta_{n,\tau}\ , \qquad (5)$$

for $n = 0,1,\ldots,[2I/M]$, where $[2I/M]$ means an integer part of $2I/M$. Conditions (5) can be written in matrix notation

$$\left(h_i^t\right)^T A_n h_k^d = \delta_{i,k}\delta_{n,\tau}\ , \qquad (6)$$

where $h_i = [h_i(0) \quad h_i(1) \quad \cdots \quad h_i(I)]^T \in \Re^{I+1}$,

$$A_n = \{a_n(m,l)\} \in \Re^{(I+1)\times(I+1)}$$

and m and l are numbers of rows and columns of matrices A_n, respectively. Elements of these matrices are given by formula

$$a_n(m,l) = \begin{cases} 1 & \text{if} \quad m+l = Mn+2 \\ 0 & \text{if} \quad m+l \neq Mn+2 \end{cases}, \quad n = 0,\ldots,[2I/M].$$

For each pair $i,k \in \{1,2,\cdots,M\}$ of filter numbers, condition (6) gives $[2I/M]+1$ equations. Therefore it results in the system of $M^2([2I/M]+1)$ equations.

Now, let us consider the specific case when filters have low orders and a transmultiplexer system has delay $\tau = 1$. Let the orders of all combining filters be equal to K and let K depend on the number of channels in the following way $K \le M - 1$. Let the order of all separation filters be $L \le M$ and moreover $h_i^s(0) = 0$ for all $1 \le i \le M$. Let us introduce two matrices

$$G^c = \begin{bmatrix} h_1^c(M-1) & h_1^c(M-2) & \cdots & h_1^c(0) \\ h_2^c(M-1) & h_2^c(M-2) & \cdots & h_2^c(0) \\ \cdots & \cdots & \cdots & \cdots \\ h_M^c(M-1) & h_M^c(M-2) & \cdots & h_M^c(0) \end{bmatrix} \qquad (7)$$

$$G^s = \begin{bmatrix} h_1^s(1) & h_2^s(1) & \cdots & h_M^s(1) \\ h_1^s(2) & h_2^s(2) & \cdots & h_M^s(2) \\ \cdots & \cdots & \cdots & \cdots \\ h_1^s(M) & h_2^s(M) & \cdots & h_M^s(M) \end{bmatrix} \qquad (8)$$

which consist of combining and separation filter coefficients, respectively. Both matrices

$$G^c, G^s \in \Re^{M\times M}$$

are square and their dimensions depend on number of channels. Under these assumptions, conditions (6) can be written in a simple form

$$G^c G^s = E \qquad (9)$$

where E is a unitary matrix and both matrices G^c and G^s must be nonsingular. Condition (9) is simple but its mathematical justification is not trivial and will be published elsewhere.

Condition (9) suggests the method for finding filter coefficients:

- choose an arbitrary matrix (7) (i.e. coefficients of composition filters)
- compute the coefficients of separation filters

$$G^s = (G^c)^{-1}. \qquad (10)$$

Until now, for the practical applications, the FIR filter bank was frequently approximated via the least-squares method [2,3,8]. Practically, the coefficients of the filters had finite precision occurrence, so some errors were not avoidable. Obviously, our proposed method performs better than the least-squares method. The strict solution can be found instead of seeking the approximate solution by iterative method. It is possible to provide all calculations without divisions and using the integer numbers only to omit the rounding errors. For this case it is convenient to use such filters that

$$\det G^c = \det G^s = 1\,.$$

One of such the simplest realizations has form

$$h_i^c(M-k) = \begin{cases} 0 & \text{for} \quad k < i \\ 1 & \text{for} \quad k = i \\ c_{ik} & \text{for} \quad k > i, \end{cases}$$

where c_{ik} are arbitrary chosen integer numbers and $i,k \in \{1,2,\cdots,M\}$. Afterwards the values of separation filters H_l^s coefficients must be computed according to (10).

It is possible to proceed in an opposite direction as well. We can assume the coefficients of separation filters (i.e. matrix (8)) and its inverse matrix will constitute matrix (7). In such way we obtain the coefficients of combining filters.

An interesting case appears under an additional assumption that separation filters are orthogonal in such sense that

$$\sum_{l=1}^{M} h_i^s(l)\, h_m^s(l) = \begin{cases} 0 & \text{for} \quad i \neq m \\ 1 & \text{for} \quad i = m. \end{cases}$$

To fulfill these requirements, filter coefficients can acquire for example the Walsh or Haar function values. That case provides a rule

$$G^c = (G^s)^T \qquad (11)$$

to design the combining filters. A very simple condition

$$h_i^c(M-k) = h_i^s(k) \quad \text{for} \quad i,k = 1,\ldots,M$$

results from (11) taking into account (7) and (8).

3. EXAMPLES

Let us consider two simple examples: a transmultiplexer which consists of two channels only and a transmultiplexer which consists of four channels. For the first case let us assume the FIR combining filters

$$h_1^c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad h_2^c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, according to (10) we compute the coefficients of separation filters

$$h_1^s = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad h_2^s = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

For the case of four channels, let us assume the following values of FIR combining filters

$$h_1^c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad h_2^c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad h_3^c = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad h_4^c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Simple computations, according to (10) give us the coefficient of separation filters

$$h_1^s = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad h_2^s = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad h_3^s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad h_4^s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

To obtain the orthogonal separation filters let us use the Walsh functions. For the case of four channels we can assume

$$h_1^s = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad h_2^s = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad h_3^s = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad h_4^s = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

and it appears that both matrices, (7) and (8), are not only orthogonal but symmetrical as well. From

$$G^s = G^c = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

we obtain immediately the orthogonal combining filters

$$h_1^c = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad h_2^c = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad h_3^c = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad h_4^c = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

The characteristics of these filters are presented in Fig. 2 and Fig. 3. For each channel the amplitude characteristics of composition and separation filters (see Fig.2) are exactly the same but their phase characteristics are different (see Fig.3). These properties result from the relationships

$$H^s(\underline{f}) = H^c(\underline{f})\, e^{-2\pi j \underline{f}}$$

obtained for the filters in the first and the forth channel and

$$H^s(\underline{f}) = -H^c(\underline{f})\, e^{-2\pi j \underline{f}}$$

obtained for the second and the third channel, where $\underline{f} \in [0, 0.5]$ is a normalized frequency.

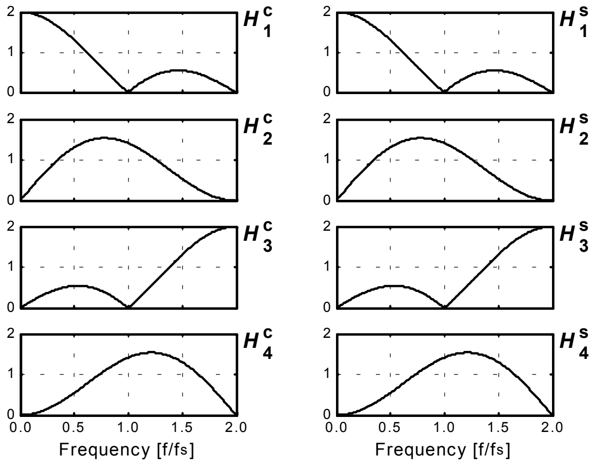

Figure 2. Amplitude characteristics of orthogonal filters: composition (on the left) and separation (on the right).

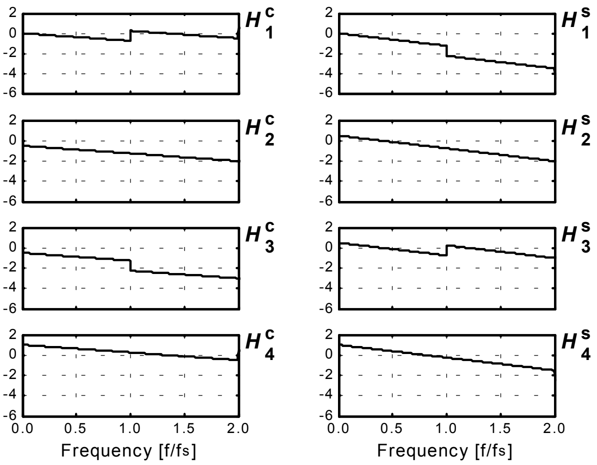

Figure 3. Phase characteristics of orthogonal filters.

The presented above examples depict the extraordinary simplicity of filters, which can be obtained while using the above described algorithm. The second example presents filters which do not need multiplications. Their coefficients are equal to -1 or 0 or 1 only. The third example has such property as well, but in that case we obtain the amplified output signals.

4. CONCLUSIONS

We have to stress the meaning of the usage of integer filters. The previous methods created real values coefficients. The perfect reconstruction condition was fulfilled only theoretically because of the finite precision of digital equipment and non-integer values of filter coefficient. Practically, output signals were a little different than input ones. For sending uncoded images or audio signals such solutions were satisfying. On the other hand, software files or coded multimedia data like MPEG files even with very slight changes are damaged and not possible to use. For example, if the first frame of MPEG video sequence is wrong the next hundred of them can be damaged as well because the first one was the Intra frame. Many other frames use it as the frame of reference. The problem of any occurring error in software files is even more crucial.

The motivation for a developing new designing method for transmultiplexer systems has resulted from the fact that quantization errors in digital filtering for real numbers are inevitable. It is trivial that imperfect filters cause the crosstalk phenomenon. The crosstalk in transmultiplexer systems can be completely eliminated by the properly designed filters. An important improvement is possible if a direct design approach based on integer number calculations is applied. If integer filters are incorporated in filter banks then not only theoretically but also in practice the perfect reconstruction conditions are applied to input signals when they have fixed complexity. Only simple and fast calculations like upsampling, downsampling, integer filtering and signal summing are carried out on-line in such transmultiplexer systems.

It is an important observation that the perfect reconstruction can be obtained without assumption that filters have the separate frequency bands. Moreover, if we compare (see Fig.2) filters H_2^c and H_4^c, it is noticeable that filter amplitude characteristics can be similar.

The considerations presented in this paper began from the assumptions that the filter orders are low and depend on the number of channels. It is possible to design the transmultiplexer system equipped with filters of higher orders but the mathematical conditions are then more complicated and it is difficult to form the algorithm of filters designing.

5. ACKNOWLEDGMENTS

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REFERENCES

[1] A.N. Akansu, P. Duhamel, X. Lin, and M. Courville, "Orthogonal Transmultiplexers in Communication: A Review," *IEEE Trans. on Signal Processing*, vol. 46, no. 4, pp. 979-995, April 1998.

[2] T.Liu, and T. Chen, "Optimal design of multi-channel transmultiplexers," *Signal Processing*, vol. 80, no. 10, pp. 2141-2149, Oct. 2000.

[3] T. Liu, and T. Chen, "Design of Multichannel Nonuniform Transmultiplexers Using General Buliding Blocks," *AIEEE Trans. on Signal Processing*, vol. 49, no. 1, pp. 91-99, 2001.

[4] M. Vetterli, "A theory of multirate filter banks," *IEEE Trans. Accoust. Speech Signal Process*, vol. 35, pp. 356-372, 1987.

[5] M. Vetterli, and J. Kovacevic, *Wavelets and Subband Coding*. Englewood Cliffs, NJ: Prentice Hall, 1997.

[6] A. J. Viterbi, *CDMA: Principles of Spread Spectrum Communication*. Addison Wesley Longman, Inc., 1995.

[7] K. S. Zigangirov, *Theory of Code Division Multiple Access Communication*. Wiley-IEEE Press, 2004.

[8] M.Ziółko, P.Sypka, and T.Kozłowski, "Application of 1-D Filters to Image Transmultiplexing," in *Proc. EUSIPCO 2004*, Vienna, September 6-10. 2005, pp.545-548.